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Extremal aiming in problems with an unknown level of dynamic disturbance☆

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ABSTRACT

The method of extremal aiming, well-known in the theory of differential games, is applied to problems in which the level of dynamic disturbance is not stipulated in advance. Problems with linear dynamics, a fixed termination time and a geometric constraint on the effective control are considered. The aim of the control is to bring the system into a specified terminal set at the instant of termination. A feedback control method is proposed which ensures successful completion if the disturbance does not exceed a certain critical level. Here, "weak" disturbance is countered by a "weak" effective control. A guarantee theorem is formulated and proved. An illustrative example is considered.

1. Introduction

Methods for solving problems in which geometric constraints on the control actions of both players are stipulated according to the formulation, are well developed in the theory of antagonistic differential games.[1–5] However, in many practical problems, a geometric constraint is only imposed on the effective control (on the control of the first player) while the imposition of such a constraint on the dynamic action of the disturbance (on the control of the second player) is unnatural. Moreover, the optimal feedback control of the first player, obtained within the limits of the standard formalization of an antagonistic differential game is directed to countering the worst disturbance. In real situations, the dynamic disturbance does not, as a rule, act in the worst way.

It is desirable to have a feedback control process which operates successfully over a wide range of disturbance. Here, the "weaker" or "less optimal" the disturbance, the "weaker" the effective control countering it must be. The aim of this paper is to propose such a method which rests on the established theory of differential games.

The problem considered is similar to problems of suppressing a bounded external disturbance by a control system, which are being intensively studied at present.[6–9] The main difference in this paper, apart from the mathematical apparatus used, lies in the fact that the control process in it is considered in a finite time interval and the effective control is constrained by a geometric limitation, according to the formulation of the problem. Among the papers, that use the results of the theory of differential games and are orientated towards problems with an unknown level of disturbance, we mention Ref. 10.

The central concept used in this paper is the concept of a stable bridge.[3–5] A set in a *time* × *phase vector* space, in which the first player, by using his control and discriminating the adversary, can maintain the motion of the system right up to the instant of termination, is called a stable bridge.

Consider a family of differential games where the geometric constraint on the second player's control depends on a scalar parameter. We will associate a certain constraint on the first player's control and a certain stable bridge with each value of the parameter. We will assume that the family of bridges is arranged in the order of increasing values of this parameter. The first player guarantees the retention of the phase vector in the tube of a stable bridge using his control, the level of which corresponds to the tube being considered if the second player's control also satisfies the corresponding constraint. The family of bridges enables us to construct the first player's feedback control and to describe the guarantee ensured by the control.

We will now explain how this takes place. Suppose a disturbance, which does not exceed a certain level, acts on a control system. The motion of the system will then intersect bridges of the family which has been constructed until the boundary of the bridge is reached (from above or below) corresponding to the level of disturbance realized. The motion will subsequently proceed within the limits of this bridge. Fine tuning (adaptation) of the level of the effective control therefore occurs automatically under a level of disturbance, unknown in advance.

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The idea of specifying an ordered family of stable bridges described above is very general. Its specific embodiment is associated with the possibility of the analytical or numerical construction of stable bridges. In the theory of differential games, there is a considerable number of publications[11–21] dealing with algorithms for the numerical construction of the most stable bridges and the sets for the level of the value function. The methods which have been developed can be used to construct the above mentioned family of stable bridges.

Problems with linear dynamics, a fixed instant of termination and a convex compact terminal set are considered in which the first player attempts to direct the system. The effective vector control is constrained by a geometric limitation in the form of a convex compact. The features which have been enumerated enable us to construct an ordered family of stable bridges and the corresponding adaptive control. Actually, in the case being investigated, it is sufficient to construct just two special maximum stable bridges in advance and store them in the computer's memory. The t-section of the appropriate stable bridge from the above mentioned family is calculated on the basis of these during the motion at a current instant of time t. The first player's control is generated using extremal aiming[3,5,22] in this section. The effectiveness of the algorithm is due to the fact that all the t-sections of the bridges in the family considered are convex.

A theorem concerning the guarantee, which is provided to the first player by the proposed control method, is formulated and proved.

At present, the algorithm is numerically implemented[23–25] for the case when the terminal set is only determined by two or three components of the phase vector at the instant of termination.

The paper is completed with an example of the modelling of a linearized problem on the encounter of two weakly manoeuvring objects. The description of the dynamics is borrowed from the publications of Shinar and his co-authors.[26,27]

The investigation touches on the analysis of the case of a scalar effective control.[24] A brief description of the adaptive control method in the case of an arbitrary compact constraint on the effective control is available.[28]

2. Formulation of the problem

Consider a linear differential game with a fixed instant of termination

$$\dot{z} = A(t)z + B(t)u + C(t)\upsilon,\ z \in R^m,\ t \in T,\ u \in P \subset R^p,\ \upsilon \in R^q \tag{2.1}$$

Here u and υ are the vector controls of the first and second players, P is the convex compact constraint on the first player's control and $T = [\vartheta_0, \vartheta]$ is the game interval. We will stipulate that the set P contains the zero of the space R^p. The matrix-valued functions A and C are continuous with respect to t and the matrix-valued function B satisfies the Lipschitz condition in the interval T. There is no specific constraint of any kind on the control υ.

The first player attempts to bring n separate components of the phase vector of system (2.1) into the terminal set M at an instant ϑ. The set M is assumed to be a convex compact in the space of the above mentioned n components of the phase vector z. We will stipulate that the set M contains a certain neighbourhood of the origin of coordinates of this space and we will adopt the origin of coordinates as the centre of the set M. It is in the interest of the first player to transfer the n separate components of the vector z as close as possible to the centre of the set M.

It is now required to propose a method for constructing of the adaptive control for system (2.1).

We will change to a system, the right-hand side of which does not contain the phase vector:

$$\dot{x} = D(t)u + E(t)\upsilon,\ x \in R^n,\ t \in T,\ u \in P \subset R^p,\ \upsilon \in R^q \tag{2.2}$$

The transition is accomplished (see Ref. 3, p. 160 and Ref. 5, pp. 89 - 91) using the relations

$$x(t) = Z_{n,m}(\vartheta, t)z(t),\ \ D(t) = Z_{n,m}(\vartheta, t)B(t),\ \ E(t) = Z_{n,m}(\vartheta, t)C(t)$$

where $Z_{n,m}(\vartheta, t)$ is a matrix composed of n rows of the fundamental Cauchy matrix for the system $\dot{z} = A(t)z$ corresponding to the larger components of the vector z in the space of which the set M is defined. The first player attempts to bring the phase vector of system (2.2) into the set M at the instant of termination ϑ.

The subsequent calculations will be carried out for system (2.2). The control $U(t, x)$ which has been constructed is applied to system (2.1) in the form $U(t, Z_{n,m}(\vartheta, t)z$.

3. System of stable bridges

The symbol $S(t) = \{x \in R^n : (t, x) \in S\}$ henceforth denotes the section of the set $S \subset T \times R^n$ at an instant $t \in T$. Suppose $O(\varepsilon) = \{x \in R^n : |x| \leq \varepsilon\}$ is a sphere of radius ε in the space R^n with its centre at zero.

3.1. Stable bridges

Consider an antagonistic differential game in the interval $T = [\vartheta_0, \vartheta]$ with a terminal set $\mathcal{M}$ and geometric constraints $\mathcal{P}$ and $\mathcal{Q}$ on the players' controls

$$\dot{x} = D(t)u + E(t)\upsilon,\ x \in R^n,\ t \in T,\ \mathcal{M},\ u \in \mathcal{P},\ \upsilon \in \mathcal{Q} \tag{3.1}$$

Here, the matrices $D(t)$ and $E(t)$ are the same as those in system (2.2). The sets $\mathcal{M}$, $\mathcal{P}$ and $\mathcal{Q}$ are assumed to be convex compacts. They are regarded as the parameters of the game.

Below, $u(\cdot)$ and $\upsilon(\cdot)$ will denote measurable functions of time with values in the sets $\mathcal{P}$ and $\mathcal{Q}$ respectively. We will denote the motion of system (3.1) (and, consequently, of system (2.2)) emerging from a point x_* at an instant t_* on account of the controls $u(\cdot)$ and $\upsilon(\cdot)$ by $x(\cdot; t_*, x_*, u(\cdot), \upsilon(\cdot))$.

Following Krasovskii and Subbotin,[3,5] we will now define the concept of stable and maximal stable bridges.

We call a set $W \subset T \cdot R^n$ a *stable set* for system (3.1) in the case of certain fixed sets $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{M}$, if $W(\vartheta) = \mathcal{M}$ and the following stability property is satisfied: for any position and for any second player's control $\upsilon(\cdot)$, the first player can choose his control $u(\cdot)$ such that the position $(t, x(t)) = (t, x(t; t_*, x_*, u(\cdot), \upsilon(\cdot)))$ remains in the set W at any instant $t \in (t_*, \vartheta]$. The maximal set with respect to inclusion $W \subset T \times R^n$, $W(\vartheta) = \mathcal{M}$, which possesses the property of stability, is called a *maximal stable bridge*.

A maximal stable bridge is[3,5] a closed set. Its t-sections are convex (Ref. 5, p. 87) on account of the linearity of system (3.1) and the convexity of the set $\mathcal{M}$.

3.2. Construction of a system of stable bridges

1°. We select a set $Q_{max} \subset R^q$, which is treated as a "maximal" constraint on the second player's control which the first player agrees to regard as "reasonable" when bringing system (2.2) into the set M. We assume that the set Q_{max} contains the zero of its space. This assumption is not burdensome since the aiming problem must be solvable when there is no disturbance. We will denote the maximal stable bridge for system (3.1), corresponding to the parameters $\mathcal{P} = P$, $\mathcal{Q} = Q_{max}$, $\mathcal{M} = M$, by W_{main}.

We additionally stipulate that the set Q_{max} is chosen such that the inclusion

$$O(\varepsilon) \subset W_{main}(t) \tag{3.2}$$

is satisfied for a certain $\varepsilon > 0$ and any $t \in T$. We will henceforth assume that the number ε is fixed.

Hence, W_{main} is a closed tube in the space $T \times R^n$ which breaks off at an instant' in the set M. Any of its t-sections $W_{main}(t)$ are convex and contain the zero of the space R^n together with a certain neighbourhood.

2°. We will now introduce a further closed tube $W_{add} \subset T \times R^n$, each section $W_{add}(t)$ of which is an attainability set of system (3.1) at an instant t with the initial set $O(\varepsilon)$ taken at the instant ϑ_0. In constructing the tube W_{add}, we assume that the first player is absent ($u \equiv 0$) and the second player's control is limited by the constraint on Q_{max}. It is easily seen that W_{add} is a maximal stable bridge for system (3.1) when

$$\mathcal{P} = \{0\},\ \mathcal{Q} = Q_{max},\ \mathcal{M} = W_{add}(\vartheta)$$

For any $t \in T$, a section $W_{add}(t)$ is convex and the following inclusion holds

$$O(\varepsilon) \subset W_{add}(t) \tag{3.3}$$

3°. We now consider the family of tubes $W_k \subset T \times R^n (k \geq 0)$, the sections $W_k(t)$ of which are defined by the relations

$$W_k(t) = \begin{cases} kW_{main}(t), & 0 \leq k \leq 1 \\ W_{main}(t) + (k-1)W_{add}(t), & k > 1 \end{cases}$$

The sets $W_k(t)$ are compact and convex. By virtue of relations (3.2) and (3.3), the strict inclusions are satisfied for any numbers $0 \leq k_1 < k_2 \leq 1 < k_3 < k_4$.

$$W_{k_1}(t) \subset W_{k_2}(t) \subset W_{k_3}(t) \subset W_{k_4}(t)$$

It has been shown[23] (see footnote [1]) that a tube W_k when $0 \leq k \leq 1$ is a maximal stable bridge for system (3.1) corresponding to a constraint kP, imposed on the first player's control, a constraint kQ_{max} imposed on the second player's control and a terminal set kM. When $k > 1$, the set W_k is a stable bridge (but not, generally speaking, a maximal stable bridge) for the parameters

$$\mathcal{P} = P,\ \mathcal{Q} = kQ_{max},\ \mathcal{M} = M + (k-1)W_{add}(\vartheta)$$

We therefore have an expanding system of stable bridges in which each large bridge corresponds to a large constraint imposed on the second player's control. This system of bridges is generated by the two bridges W_{main} and W_{add} using the operations of algebraic summation and multiplication by a non-negative numerical parameter.

We put

$$P_k = \min\{k, 1\} \cdot P, \quad k \geq 0$$

The function $V : T \times R^n \to R$ is given in the form

$$V(t, x) = \min\{k \geq 0 : (t, x) \in W_k\}$$

Relations (3.2) and (3.3) ensure that the Lipschitz condition with a constant $\lambda = 1/\varepsilon$ is satisfied for the function $x \mapsto V(t, x)$ for each $t \in T$.

4. Adaptive feedback control

The adaptive control $(t, x) \mapsto U(t, x)$ is constructed in the following way.

The number $\xi > 0$ is fixed.

We now consider an arbitrary position (t, x). In the case when $|x| > \xi$, we find the number k^* defining the bridge W_{k^*}, the section $W_{k^*}(t)$ of which stands at a distance ξ from the point x. On the boundary of the set $W_{k^*}(t)$, we calculate the point x^* which is closest to x. We have $|x^* - x| = \xi$. The vector $u^* \in P_{k^*}$ is prescribed from the extremum condition

[1] Also, see: Ganebnyi SA, Kumkov SS, Patsko VS, Pyatko SG. Robust control in game problems with linear dynamics. *Preprint. Ekaterinburg:IMM UrO Ross Akad Nauk*;2005.

$$(x^* - x)' D(t) u^* = \max\{(x^* - x)' D(t) u : u \in P_{k^*}\} \tag{4.1}$$

We put $U(t, x) = u^*$ and, if $|x| \le \xi$, we take $U(t, x) = 0$.

The formation of the control U in terms of the function V is written in the following way. In a closed ξ-neighbourhood of the point x, we find the point x^* of the minimum of the function $V(t, \cdot)$. We put $k^* = V(t, x^*)$. We choose the control $U(t, x)$ from the set P_{k^*} using relation (4.1). We shall call the point x^* the aiming point.

We use the control U in a discrete scheme[3,5,22] with a step size $\Delta > 0$ in time. The control is chosen at the initial instant when the length Δ is next sampled. The control remains constant up to the end of the sampling.

The control (strategy) U is therefore formed on the basis of the rule of extremal aiming,[3,5,22] which is widely known in the theory of differential games. In this paper, it was modernized for a problem in which there is no a priori geometric constraint on the second player's control.

When the strategy U is used, there is automatic fine tuning of the level of the first player's control under the level of the second player's control (dynamic disturbance) which is realized. Actually, in the case of a "weak" disturbance, the motion proceeds within the system of stable bridges and, correspondingly, the current index k^*, defining the bridge of aim, decreases. On the other hand, in the case of "strong" disturbance, the current index k^* increases. The change in this index has a direct influence on the change in the current level of the constraint on the first player's control from which, by virtue of the strategy U, the extremal control is chosen.

We now present formulae for calculating of the value of k^* at the instant t when $|x| > \xi$. The values of the support functions of the sections $W_{\text{main}}(t)$ and $W_{\text{add}}(t)$ on a vector $l \in R^n$ are denoted by $\rho_{\text{main}}(l)$ and $\rho_{\text{add}}(l)$.

We put

$$\tilde{k} = \max_{|l|=1} \frac{l'x - \xi}{\rho_{\text{main}}(l)}, \qquad \bar{k} = 1 + \max_{|l|=1} \frac{l'x - \rho_{\text{main}}(l) - \xi}{\rho_{\text{add}}(l)}$$

We have

$$k^* = \begin{cases} \tilde{k}, & \tilde{k} \le 1 \\ \bar{k}, & \tilde{k} > 1 \end{cases} \tag{4.2}$$

Proof of relation (4.2). In fact, suppose $\tilde{k} \le 1$. The inequalities

$$l'x \le \tilde{k}\rho_{\text{main}}(l) + \xi, \quad \forall l, \quad |l| = 1$$

are then satisfied.

This means that the point x is at a distance not exceeding ξ from the set $\tilde{k}W_{\text{main}}(t) \subset W_{\text{main}}(t)$. Consequently, the unknown $k^* \le 1$. In this case, k^* is the smallest index $k \in [0, 1]$ for which

$$l'x \le k\rho_{\text{main}}(l) + \xi, \quad \forall l, \; |l| = 1$$

For a fixed l, $|l| = 1$, the smallest value of $k(l)$ is described by the formula

$$k(l) = \frac{l'x - \xi}{\rho_{\text{main}}(l)}$$

Hence,

$$k^* = \max_{|l|=1} k(l) = \tilde{k}$$

Now, suppose $\tilde{k} > 1$. This means that the point x does not belong to the closed ξ-neighbourhood of the set $W_{\text{main}}(t)$. Consequently, the unknown $k^* > 1$. In this case, k^* is the smallest index $k > 1$ for which

$$l'x \le \rho_{\text{main}}(l) + (k-1)\rho_{\text{add}}(l) + \xi, \quad \forall l, \; |l| = 1$$

Hence, $k^* = \bar{k}$.

5. A guarantee theorem

We will now present a theorem describing the guarantee to the first player when the proposed method of adaptive control is used.

We put

$$d = \max_{t \in [\vartheta_0, \vartheta]} \max_{u \in P} |D(t)u|$$

As the Lipschitz constant β of the mapping $t \mapsto D(t)$, we take the maximum of the Lipschitz constants of the functions $t \mapsto D_j(t)$, where $D_j(t)$ is the j-th column of the matrix $D(t)$, $j = 1, \ldots, p$.

Suppose κ is the maximum of the deviations along the coordinates of the points of the set P from zero:

$$\kappa = \max_j \max\{|u_j| : u \in P\}$$

Theorem. *Suppose $\xi > 0$ and U is the adaptive strategy of the first player which is extremally aimed in the case of a specified distance ξ. We choose arbitrary $t_0 \in T$, $x_0 \in R^n$ and $\Delta > 0$ and assume that the second player's control $\upsilon(\cdot)$ in the interval $[t_0, \vartheta]$ will be bounded by the set $c^* Q_{max}$, $c^* \geq 0$. We use the notation*

$$s^* = \max\{c^*, V(t_0, x_0)\}$$

Suppose $x(\cdot)$ is the motion of system (2.2), which emerges at the instant t_0 from the point x_0 generated by the strategy U in the discrete scheme with a step size Δ and by the second player's control $\upsilon(\cdot)$. The realization $u(t) = U(t, x(t))$ of the first player's control is then subject to the inclusion

$$u(t) \in \min\{s^* + E(t, t_0, \Delta, \xi), 1\} \cdot P, \quad t \in [t_0, \vartheta] \tag{5.1}$$

At the same time, the value $V(t, x(t))$ of the function V satisfies the inequality

$$V(t, x(t)) \leq s^* + E(t, t_0, \Delta, \xi) + \lambda\xi, \quad t \in [t_0, \vartheta] \tag{5.2}$$

Here,

$$E(t, t_0, \Delta, \xi) = \lambda\Delta(t - t_0)\left(p\beta\kappa + \frac{4d^2 + (p\beta\kappa\Delta)^2}{2\xi}\right) + 2\lambda d\Delta$$

Proof. Suppose $x^*(t)$ is the aiming point for the current point $x(t)$. In order to obtain estimates (5.1) and (5.2), it is sufficient to prove the estimate

$$V(t, x^*(t)) \leq s^* + E(t, t_0, \Delta, \xi) \tag{5.3}$$

Inclusion (5.1) follows from estimate (5.3) when account is taken of the rule of extremal aiming for an adaptive control and the fact that the relation $t \mapsto E(t, t_0, \Delta, \xi)$ is monotonically increasing. Inequality (5.2) follows from the Lipschitz character of the function V with respect to x and the fact that the points $x(t)$ and $x^*(t)$ are separated by a distance not greater than ξ.

1) Suppose a constant control u^* of the first player, generated at an instant t_* from the aiming condition, acts in a certain interval $[t_*, t_* + \delta]$. Then, $x^*(t_*)$ is the point of the set $W_{k^*}(t_*)$ which is closest to $x(t_*)$. We have $|x^*(t_*) - x(t_*)| \leq \xi$. The unit aiming vector $l(t_*)$ is directed from the point $x(t_*)$ to the point $x^*(t_*)$ (Fig. 1*a*).

We will assume that the disturbance (the second player's control) does not exceed a level of $x^* Q_{max}$ corresponding to the bridge W_{k^*}. Suppose $\upsilon(\cdot)$ is the realization of the disturbance in $[t_*, t_* + \delta)$. We will denote the position, at the instant $t_* + \delta$, of the motion from the point $x(t_*)$ on account of the constant control u^* and the disturbance $\upsilon(\cdot)$ by the symbol e. We choose the function $u(\cdot)$, which is measurable with respect to $\upsilon(\cdot)$, with values on the set P_{k^*} from the condition for the stability of the set W_{k^*} such that the corresponding motion from the point $x^*(t_*)$ in the interval $[t_*, t_* + \delta)$ goes through the sections $W_{k^*}(t)$. Suppose b is the position of this motion at the instant $t_* + \delta$.

We will now estimate the distance $r(\delta)$ between the points e and b.

We consider an auxiliary motion from the point $x(t_*)$ which is a copy of the motion on account of the stability from the point $x^*(t_*)$. We will denote the position of this motion at an instant $t_* + \delta$ by a.

We distinguish three cases:

$$l'(t_*)e > l't_*)b, \quad l'(t_*)b \geq l'(t_*)e \geq l'(t_*)a, \quad l'(t_*)e < l'(t_*)a$$

In the first of these, we have

$$r(\delta) = |e - b| \leq |e - a| \leq 2d\delta$$

In this estimate, account has been taken of the fact that the points e and a are generated by motions with the one and the same initial state and the one and the same control $\upsilon(\cdot)$.

In the second case,

$$r(\delta) \leq \sqrt{\xi^2 + (2d\delta)^2}$$

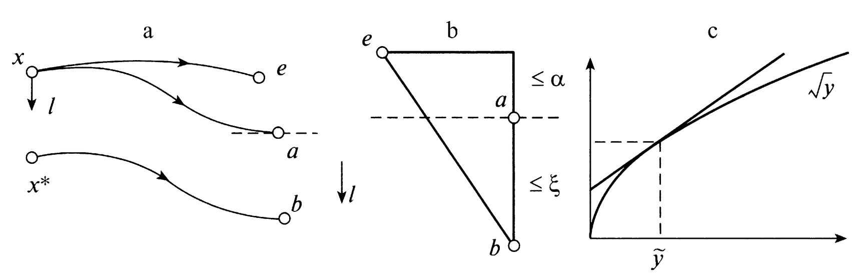

Fig. 1.

For the third case, we write

$$l'(t_*)(a-e) = l'(t_*)\left(\int_{t_*}^{t_*+\delta} D(t)u(t)dt - \int_{t_*}^{t_*+\delta} D(t)u^*dt\right) = I_1 + I_2$$

$$I_1 = l'(t_*)\int_{t_*}^{t_*+\delta} D(t_*)(u(t)-u^*)dt, \quad I_2 = l'(t_*)\int_{t_*}^{t_*+\delta}(D(t)-D(t_*))(u(t)-u^*)dt$$

Since the vector u^* was chosen from the extremal aiming condition using the vector $l(t_*)$, then $I_1 \le 0$ and, therefore,

$$l'(t_*)(a-e) \le I_2$$

We then have

$$I_2 = l'(t_*)\int_{t_*}^{t_*+\delta}\sum_{j=1}^{p}(D_j(t)-D_j(t_*))(u_j(t)-u_j^*)dt$$

The expression on the right is bounded from above by the quantity $\alpha(\delta) = p\beta\kappa\delta^2$. As a result,

$$l'(t_*)(a-e) \le \alpha(\delta)$$

Consequently, the distance $r(\delta)$ between the points e and b in the third case satisfies the inequality (see Fig. 1b)

$$r(\delta) \le \sqrt{(\xi+\alpha(\delta))^2 + (2d\delta)^2} \tag{5.4}$$

Comparing the estimates for each of the three cases, we conclude that estimate (5.4) can be taken as a universal estimate.

2) We now replace estimate (5.4) in the form of the square root by a more convenient linear estimate, and, in fact, the estimate

$$\sqrt{y} \le \sqrt{\tilde{y}} + \frac{1}{2\sqrt{\tilde{y}}}(y-\tilde{y}) \tag{5.5}$$

holds for all $\tilde{y} > 0$ and $y \ge 0$ (Fig. 1b).

In the case considered, we take $\tilde{y} = \xi^2$. By virtue of estimates (5.4) and (5.5), we obtain the estimate of the distance from the point $x(t_* + \delta)$ to the section $W_{k^*}(t_* + \delta)$ at the instant $t_* + \delta$

$$r(\delta) \le \xi + \frac{1}{2\xi}((\xi+\alpha(\delta))^2 + (2d\delta)^2 - \xi^2) = \xi + \delta^2\eta(\delta) \tag{5.6}$$

where

$$\eta(\delta) = p\beta\kappa + \frac{(p\beta\kappa\delta)^2}{2\xi} + \frac{2d^2}{\xi}$$

3) We calculate the minimum of the function $V(t_* + \delta, \cdot)$ in a sphere of radius ξ with centre at the point $x(t_* + \delta) = e$. Suppose f is the minimum point.

We will assume that the point $b \in W_{k^*}(t_* + \delta)$, which was discussed in Subsection 1, is located outside a sphere of radius ξ with its centre at the point e. Suppose h is a point on the boundary of this sphere belonging to the segment be. On the basis of estimate (5.6), we conclude that

$$|h-b| = r(\delta) - \xi \le \delta^2\eta(\delta)$$

We have

$$V(t_*+\delta, f) \le V(t_*+\delta, h) \le V(t_*+\delta, b) + \lambda\delta^2\eta(\delta) \le V(t_*, x^*(t_*)) + \lambda\delta^2\eta(\delta)$$

Here, account has been taken of the fact that

$$V(t_*+\delta, b) \le V(t_*, x^*(t_*))$$

Suppose the point b lies in a sphere of radius ξ with its centre at the point e. Then,

$$V(t_*+\delta, f) \le V(t_*+\delta, b) \le V(t_*, x^*(t_*))$$

Hence, the increment in the function V for the aiming point is estimated by the inequality

$$V(t_*+\delta, x^*(t_*+\delta)) \le V(t_*, x^*(t_*)) + \lambda\delta^2\eta(\delta) \tag{5.7}$$

4) Suppose t is an arbitrary instant from the interval $[t_0, \vartheta]$. To prove inequality (5.3), we estimate the change in the function $\tau \to V(\tau, x^*(\tau))$ in the interval $[t_0, t]$. If $V(t, x^*(t)) \le s^*$ at the instant t being considered, inequality (5.3) is obviously satisfied. We shall next

assume that $V(t, x^*(t)) > s^*$ and denote the instant of the last arrival of the point $x^*(\tau)$ in the interval $[t_0, t)$ at the level s^* of the function V by $\tilde{t}$.

When a discrete control scheme is used by the first player, the control is chosen at the initial instant of the next sampling of the length Δ. We call these instants discrete instants. We will assume that there is just one discrete instant in $(\tilde{t}, t)$ We will denote the discrete instant closest on the right to $\tilde{t}$ (closest on the left to t) by the symbol $\bar{t}$ ($\hat{t}$ respectively).

Full samplings proceed in the interval $[\bar{t}, \hat{t}]$. There are $(\hat{t} - \bar{t})/\Delta$ of them. Taking account of inequality (5.7), we obtain the estimate

$$V(\hat{t}, x^*(\hat{t})) \le V(\bar{t}, x^*(\bar{t})) + \lambda(\hat{t} - \bar{t})\Delta\eta(\Delta) \tag{5.8}$$

In $[\hat{t}, t]$, again taking account of inequality (5.7), we have the estimate

$$V(t, x^*(t)) \le V(\hat{t}, x^*(\hat{t})) + \lambda(t - \hat{t})^2\eta(t - \hat{t}) \le V(\hat{t}, x^*(\hat{t})) + \lambda(t - \hat{t})\Delta\eta(\Delta) \tag{5.9}$$

It remained to estimate the increment in $V(\tau, x^*(\tau))$ in the interval $[\tilde{t}, \bar{t}]$.

At the instant $\tilde{t}$, we have $V(\tilde{t}, x^*(\tilde{t})) = s^*$. According to the condition of the theorem, the disturbance level $\upsilon(\cdot)$ is not greater than $c^* \le s^*$. Using the function $\upsilon(\cdot)$, we find a control $u_{st}(\cdot)$ such that the motion $x_{st}(\cdot)$, on account of $u_{st}(\cdot)$, $\upsilon(\cdot)$, goes from the initial point $x^*(\tilde{t})$ in $[\tilde{t}, \bar{t}]$ through the bridge W_{s^*}. We obtain $V(\bar{t}, x_{st}(\bar{t})) \le s^*$.

The divergence of the two motions has the form of the equality

$$x(\bar{t}) - x_{st}(\bar{t}) = x(\tilde{t}) - x^*(\tilde{t}) + \int_{\tilde{t}}^{\bar{t}} D(\tau)(u(\tau) - u_{st}(\tau))d\tau$$

the modulus of the integral in which does not exceed $2d(\tilde{t} - \bar{t})$.

Consequently,

$$|x(\bar{t}) - x_{st}(\bar{t})| \le |x(\tilde{t}) - x^*(\tilde{t})| + 2d(\tilde{t} - \bar{t}) \le \xi + 2d(\tilde{t} - \bar{t}) \tag{5.10}$$

Suppose the point $x_{st}(\bar{t})$ lies outside a sphere of radius ξ with its centre at the point $x(\bar{t})$. We consider a point h on the boundary of the sphere belonging to a segment containing the points $x(\bar{t})$ and $x_{st}(\bar{t})$. Taking account of inequality (5.10), we obtain

$$V(\bar{t}, x^*(\bar{t})) \le V\bar{t}, h) \le V(\bar{t}, x_{st}(\bar{t})) + 2\lambda d(\bar{t} - \tilde{t}) \le s^* + 2\lambda d\Delta$$

Suppose the point $x_{st}(\bar{t})$ belongs to a sphere of radius ξ with its centre at the point $x(\bar{t})$. Then,

$$V(\bar{t}, x^*(\bar{t})) \le V(\bar{t}, x_{st}(\bar{t})) \le s^*$$

The estimate

$$V(\bar{t}, x^*(\bar{t})) \le s^* + 2\lambda d\Delta \tag{5.11}$$

therefore always holds.

Collecting estimates (5.8), (5.9) and (5.11), we obtain

$$V(t, x^*(t)) \le s^* + 2\lambda d\Delta + \lambda(t - \bar{t})\Delta\eta(\Delta)$$

Taking account of the fact that

$$E(t, t_0, \Delta, \xi) = \lambda(t - t_0)\Delta\eta(\Delta) + 2\lambda d\Delta$$

we arrive at estimate (5.3).

Now suppose there are no discrete instants in the interval $(\tilde{t}, t)$. Then, the estimate

$$V(t, x^*(t)) \le s^* + 2\lambda d\Delta$$

holds for an instant t which is similar to estimate (5.11). Estimate (5.3) is obviously satisfied.

6. Example

Suppose the motion of two objects in a plane is described by the relations

$$\ddot{\mathbf{x}} = F, \quad \dot{F} = (u - F)/\tau, \quad \ddot{\mathbf{y}} = \upsilon$$
$$t \in [0, \vartheta], \quad \mathbf{x}, \mathbf{y} \in R^2, \quad u \in P, \quad \upsilon \in R^2 \tag{6.1}$$

Here, $\mathbf{x}$ is the vector of the position of the first player, $\mathbf{y}$ is the vector of the position of the second player and τ is a time constant which characterizes the delay in processing the first player's control u. The geometric constraint P, imposed on the control u, has been specified. It also specifies the maximum possibilities of an acceleration F (at the initial instant $t_0 = 0$, $F(0) = 0$). The second player's control υ directly determines his acceleration. No geometric constraint of any kind is imposed on the control υ.

System (6.1) was borrowed from the publications of Shinar and his coworkers.[26,27] It arises as a result of linearization of a non-linear system in a spatial problem of aerial interception and subsequent projection of the dynamics onto a plane orthogonal to the direction of the nominal line of sight. The moment of an encounter in the nominal paths is taken as the instant of termination ϑ. Only correcting controls are used during the homing process and, on account of this, the objects are only weakly manoeuvring.

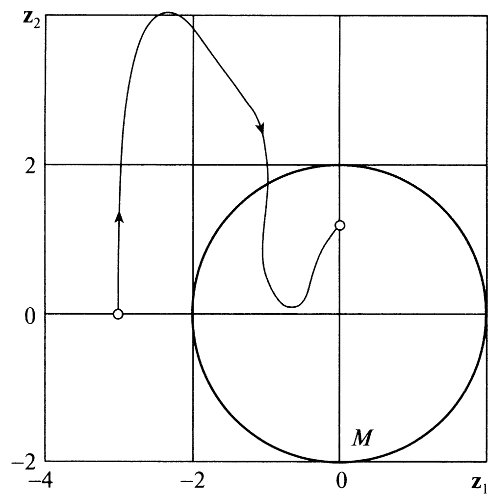

Fig. 2.

System (6.1) has been investigated[26,27] as an antagonistic game with elliptic constraints P and Q imposed on the players' controls. The parameters of the ellipses depend on the geometry of the nominal motions in the interception problem. The first player tries to reduce the miss $|\mathbf{x}(\vartheta)-\mathbf{y}(\vartheta)|$, while the interests of the second player are the opposite of this. In the above mentioned papers, the structure of the value function level sets in a differential game with the dynamics (6.1) was investigated in detail for different versions of the elliptic constraints P and Q. Three-dimensional representations of the level sets have been presented.[29]

In this paper, we abandon the a priori specification of a geometric constraint on the second player's control.

We take the initial data for problem (6.1) in the form

$$\vartheta = 10\,\text{s}, \quad \tau = 1\,\text{s}, \quad P = \left\{u \in R^2 : \frac{u_1^2}{0.87^2} + \frac{u_2^2}{1.3^2} \le 1\right\}$$

We take the radius of the terminal circle M in the difference coordinates $\mathbf{z} = \mathbf{x} - \mathbf{y}$ as being equal to 2m.

In order to construct the adaptive control, it is required that the set Q_{max} be specified, which is treated as an assumed (tentative) constraint on the second player's control. We choose it in the form

$$Q_{max} = \left\{v \in R^2 : \frac{v_1^2}{0.71^2} + \frac{v_2^2}{1.0^2} \le 1\right\}$$

We finally specify the parameter $\xi = 0.01$ m in the extremal aiming procedure and a step size $\Delta = 0.01$ with a discrete control scheme.

The initial phase vector in the difference coordinates is

$$\mathbf{z}_0 = \mathbf{x}_0 - \mathbf{y}_0 = (-3\,\text{м}, 0\,\text{м}), \quad \dot{\mathbf{z}}_0 = \dot{\mathbf{x}}_0 - \dot{\mathbf{y}}_0 = (0\,\text{m/s}, 2\,\text{m/s}), \quad F_0 = 0$$

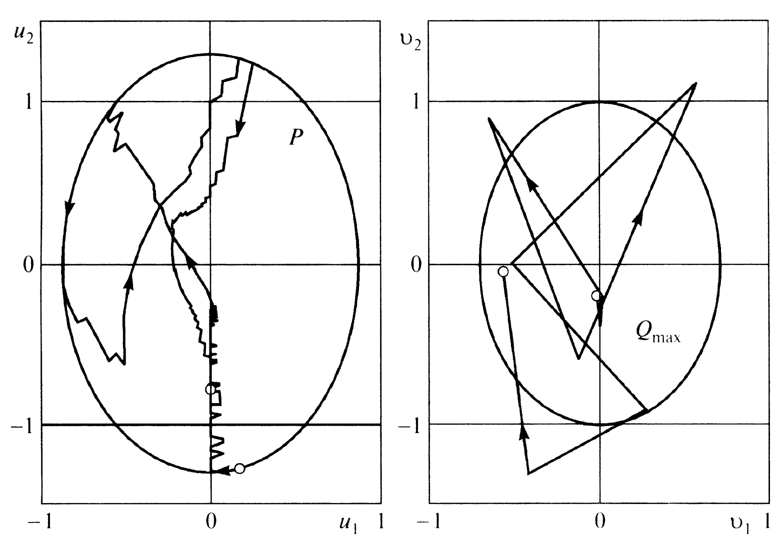

Fig. 3.

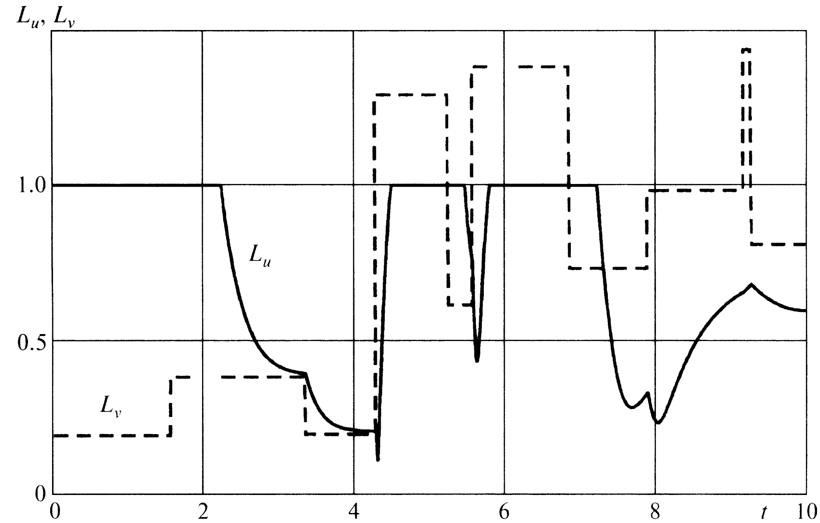

Fig. 4.

In the modelling, the second player's control was constructed as a piecewise-constant function, the vector values of which lie in the ellipse $1.5Q_{max}$ and remain constant in random time intervals with a length not greater than 3 s. The procedure for the random choice of the control υ is as follows. Initially, an angle in a range of $[0, 2\pi)$ is read off from a uniform distribution data unit. A point on the radius vector, joining the origin of coordinates to the boundary of the ellipse in the direction of the specified angle, is then uniformly chosen. The results of the modelling are shown in Figs. 2–4.

The phase trajectory in the difference coordinates $\mathbf{z}_1$ and $\mathbf{z}_2$ is shown in Fig. 2. The initial and final points are denoted by the small circles. The terminal circle M is also shown.

Hodographs of the realization of the adaptive control and the disturbance are shown in Fig. 3. The hodograph of the control u lies in the ellipse P. The hodograph of the disturbance υ falls outside the ellipse Q_{max} in some time intervals.

Graphs of the levels L_u and L_υ of the realizations of the vector control u with respect to the ellipse P (the solid curve) and of the disturbance vector υ with respect to the ellipse Q_{max} (the dashed line) are shown in Fig. 4. The choice of the control u on the boundary of the ellipse P corresponds to the value $L_u = 1$. Three intervals of the maximum level of the control u are evident: one at the beginning of the process, when the initial deviation is depleted, and two others, in the middle of the process, when the disturbance considerably exceeds the level $L_\upsilon = 1$ corresponding to the constraint Q_{max}. Outside these three intervals, the level of the effective control u is considerably below its maximum value.

Returning to Fig. 2, we note that, in spite of the fact that the disturbance realized surpassed the planned level Q_{max}, the miss at the end of the approach process is small.

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